AMS 261: Midterm 1 Review

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1 Functions of Two and Three Variables

1.1 Functions of two variables

- Graphs of a **function** have only 1 z-value, graph of **equation** can have multiple z-values.
- Slope of a contour diagram is $-b/c$
- Can estimate slopes from tables too. $m = \Delta z / \Delta x$
- $f(x, y) = a + bx + cy$
- Linear function of two variables is a plane.

1.2 Functions of three variables

1.3 Level Surfaces

$f(x, y)$ is the member of a family of level surfaces iff

$$g(x, y, z) = f(x, y) - z = 0$$

Basically, you must be able to solve for z.

2 Linear Functions

2.1 Slopes and a Point

With a point (x_0, y_0, z_0)

$$z = z_0 + b(x - x_0) + x(y - y_0)$$

$$ax + by + cz = d$$

3 Limits

Just figure it out intuitively.

4 Vectors

4.1 Displacement Vectors

For problems with Velocity, Acceleration, Force:

Speed is the sum of two vectors.

4.2 Dot Product

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$$

Two vectors are orthogonal iff

$$\vec{v} \cdot \vec{w} = 0$$

4.3 Normal Vector

Normal vectors are perpendicular to the plane. For a plane: $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$

$$\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$$

Do displacements from a common point: Then $0 = n \cdot p$

$$\vec{n} = v_1 \times v_2$$

when two vectors on a plane. $\vec{p} = [x-a, y-b, z-c]$ relative to that point (a, b, c)

4.4 Projections

$\vec{u}$ must be a unit vector.

$$\vec{v}_{parallel} = (\|\vec{v}\| \cos\theta)\vec{u} = (\vec{v} \cdot \vec{u})\vec{u}$$

$$\vec{v}_{perp} = \vec{v} - \vec{v}_{parallel}$$

In physics, $W = \vec{F} \cdot \vec{d} = \|\vec{F}_{parallel}\|\|\vec{d}\|$

4.5 Cross Product

$$\vec{v} \times \vec{w} = A \cdot \vec{n} = (\|\vec{v}\|\|\vec{w}\| \sin\theta)\vec{n}$$

$(\|\vec{v}\|\|\vec{q}\| \sin\theta)$ is the area of a parallelogram with those edges.

$$A = \|\vec{v} \times \vec{w}\|$$

4.6 Vector Algebra

$\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$
$(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda\vec{w})$
$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

4.7 Shapes

Volume of a parallelepiped:

$$V = (\vec{v} \times \vec{w}) \cdot \vec{a} = abs\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right)$$

$$V = \|\vec{v} \times \vec{w}\|$$

Volume of tetrahedron: $\frac{1}{6}$ that of parallelepiped.

5 Partial Derivatives

5.1 Formal definition

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Or, we say

$$f_x(x,y) = \frac{\delta z}{\delta x}$$

$$f_y(x,y) = \frac{\delta z}{\delta y}$$

At a point:

$$f_x(x,y) = \frac{\delta z}{\delta x}|_{a,b}$$

5.2 The Tangent Plane

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Also called a local linearization.

For an arbitrary number of variables, $f(a,b,c,\ldots) \approx f(a,b,c,\ldots) + f_a(a,b,c,\ldots) + \ldots$

5.3 The differential

$$df = f_x dx + f_y dy$$

5.4 Directional Derivative

If $\vec{u}$ is a unit vector:

$$f_{\vec{u}}(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

Geometric meaning: Slope of the cross section of a graph with plane $u_2(x-a) - u_1(y-b) = 0$

Easier way: If $\vec{u}$ is a **unit vector**:

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

Notice:

$$f_{\vec{u}}(a,b) = \nabla f(a,b) \cdot \vec{u} = \|\nabla f(a,b)\| \|\vec{u}\| \cos\theta$$

5.5 Gradient Vector

$$\text{grad} f(a,b) = \nabla f = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$$

Significance:

$$\max f_{\vec{u}}(a,b) = \|\nabla f(a,b)\|$$

$$\min f_{\vec{u}}(a,b) = -\|\nabla f(a,b)\|$$

$$\min \|f_{\vec{u}}(a,b)\| = 0$$

∇f points in direction f increases the most.
Gradient is perpendicular to level surface of $f(x,y,z)$ at a point.

5.6 Tangent to Level Surface

For $z = f(x,y)$, put into form $f(x,y) - z = 0$.